Lorentz transformation Eqution :→

Let us consider S and S′ be the two frames of references where S′ is moving with uniform velocity v relative to S. Let the observers in two systems be situated at the origin O and O′ and they want to study at P. According to O the co-ordinate of P is x, y, z and t and according to O′ it is x', y', z' & t'. Let the time is counted when O and O′ momentarily coincide and the light cap signal is produced at zero time. Consider for the eqution transmitted signal in x-direction in positive direction, for the observer O be
$x - ct = 0$. where c = speed of light and for the observer O eqn. be $x' - ct' = 0$

for the same event to satisfy both the eqn

$$x' - ct' = \lambda(x - ct) \quad \text{---} \quad (1)$$ when λ is a constt.

Similarly, in the −ve direction of x

$$x' + ct' = \mu(x + ct) \quad \text{---} \quad (2)$$ 'μ' is another constt.

Adding eqn. (1) & (2)

$$x' = \frac{\lambda + \mu}{2}x - \frac{\lambda - \mu}{2}ct$$

putting $\frac{\lambda+\mu}{2} = a$ and $\frac{\lambda - \mu}{2} = b$

$$\therefore x' = ax - bct \quad \text{---} \quad (3)$$

substracting (1) for (2)

$$ct' = act - bx$$

ie. $$t' = at - \frac{bx}{c} \quad \text{---} \quad (4)$$

The relative velocity v of the system S′ with respect to S can be obtained by putting $x' = 0$ in eqn (3)

$$x = \frac{bc}{a}t$$

or, $$\frac{x}{t} = \frac{bc}{a}$$

This quantity $\frac{x}{t}$ gives the velocity ~~v~~ of s' with respect to s. Since the system have travelled $x = vt$ distance.

$$\therefore v = \frac{x}{t} = \frac{bc}{a} \quad \text{--- (5)}$$

for $t = 0$ from eqn (1)

$$x' = ax$$

or, $\Delta x' = \Delta x \cdot a$

or, $\Delta x' = 1$

$$\therefore \Delta x = \frac{1}{a} \quad \text{--- (6)}$$

Hence two points separated by a distance $\Delta x' = 1$ as measured in system s' will appear s to have a separation $\Delta x = \frac{1}{a}$

for $t = 0$ from eqn (4)

$$act = bx$$

$$\therefore t = \frac{bx}{ac} = \frac{av}{c} \cdot \frac{x}{ac} \quad \left[\because b = \frac{av}{c} \text{ from eqn 5}\right)$$

$$\therefore t = \frac{vx}{c^2}$$

Putting this value of t in eqn. (1)

$$x' = ax - \frac{av}{c} \cdot c \cdot \frac{vx}{c^2} = a\left(1 - \frac{v^2}{c^2}\right)x$$

$$\Delta x' = a\left(1 - \frac{v^2}{c^2}\right)\Delta x \quad \text{if } \Delta x = 1$$

$$\Delta x' = a\left(1 - \frac{v^2}{c^2}\right) \quad \text{--- (7)}$$

This means that two points separated by a distance ~~$\Delta x = 1$~~ $\Delta x = 1$ have a separation

$$\Delta x' = a\left(1 - \frac{v^2}{c^2}\right)$$

on the principle of equivalence, when the separation is judged for s', the two values of separation derived above must be equal and hence

$$\frac{1}{a} = a\left(1-\frac{v^2}{c^2}\right) \quad \text{or,} \quad a^2 = \frac{1}{1-\frac{v^2}{c^2}}$$

$$\therefore \; a = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad --- (8)$$

Putting this value of 'a' in eqn (3) & (4), we get

$$x' = \frac{x}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \cdot \frac{v}{c} \cdot c \cdot t$$

$$\text{or,} \; x' = \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}} \quad --- (9)$$

$$\text{and} \; t' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \cdot t - \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \cdot \frac{v}{c} \cdot \frac{x}{c}$$

$$\text{or,} \; t' = \frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \quad --- (10)$$

In the general case where light signal is not restricted to travel along x-axis alone, the above relations are further supplemented by $y = y'$ and $z = z'$ since y and z-axis of the two systems in relative uniform motion are parallel. Thus we get the fundamental eqns of relativity

$$x' = a(x-vt), \quad y' = y, \quad z' = z$$

$$\therefore \; t' = a.\left(t-\frac{vx}{c^2}\right) \quad --- (11)$$

This is known as Lorentz transformation equations. On solving the eqn (11) we can get

$$x = a(x' + vt)$$

$$y = y'$$

$$z = z'$$

$$t = a\left(t + \frac{vx'}{c^2}\right) \quad \text{--- (12)}$$

→ Expression for Time Dilations from Lorentz transformation :→ (Time Dilations)

Let us consider two frames of references S and S'. An observer in an aeroplane is travelling with a uniform velocity V along the direction of x-axis in frame S'. The observer in the other frame S is at rest finds that the journey takes a time t as measured according to his own time scale. To find the time t' for the same journey for the observer on the plane.

$$t' = a\left(t - \frac{vx}{c^2}\right)$$

Putting $x = vt$, since it is the distance travelled by the plane for the observer at rest in time t

$$\therefore t' = a\left(t - \frac{v^2 t}{c^2}\right) = at\left(1 - \frac{v^2}{c^2}\right)$$

$$\text{or, } t' = t\left[1 - \frac{v^2}{c^2}\right]^{1/2} \quad \because a = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$$

$$\therefore t' < t.$$

Hence there are two times, one for the observer at rest and one for the observer in motion. Time for the same

Journey is lesser for the observer in motion than for the observer at rest.

These consequences of the theory concerning time which are ordinarily alluded to as time dilation i.e. time cannot be measured absolutely by only relatively. Lorentz in his transformation eqn. has showed that $t' \neq t$.

If $v = c$, $a = \infty$. in which to the observer at rest the movement of the plane will appear infinitely slow. Since the value of a reaches infinity for $v = c$, it is not possible for v to be greater than c i.e. no body can move with the velocity greater than the velocity of light.

→ Relativistic formula for composition of velocities :→

Let us consider two system S and S'. Let the body has moved through a distance dx in time dt in the system S and through dx in time dt in system S' then

$$\frac{dx}{dt} = u, \quad \frac{dx'}{dt'} = u'$$

from the transformation equation

$$x = a(x' + vt')$$

$$t = a\left(t' + \frac{vx'}{c^2}\right)$$

on differentiation,

$$dx = a(dx' + udt')$$

$$\text{and } dt = a\left(dt' + \frac{udx'}{c^2}\right)$$

$$\therefore \frac{dx}{dt} = \frac{dx' + vdt}{dt' + \frac{vdx'}{c^2}}$$

$$= \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2} \cdot \frac{dx'}{dt'}}$$

i.e. $u = \dfrac{u' + v}{1 + \dfrac{u'v}{c^2}}$

This is relativistic law of addition of velocities.

→ Q→ Show that the velocities of light is an absolute constant independence of motion of system of reference.

Ans→ The relativistic law of addition of velocities

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

Leads to a very important conclusion about the velocities of light c. In the above relation in order that u may be real u' and v must be both less than c. If we put $u' = c$ then

$$u = \frac{v + c}{1 + \frac{vc}{c^2}} = \frac{v + c}{\frac{v + c}{c}} = c$$

This means that if the velocity of the body u' as measured in S' is the velocity of light c the velocity of the same particle as measured in S is also c. Hence the velocity of light is an absolute constant independent of the velocities of system of reference.

→ Velocity of light is fundamental velocity due to:

(i) It is constant in all direction.

(ii) It is the same for all observers irrespective of velocities of source or the observer.

(iii) It is invariant of the two systems.

(iv) The addition of any velocity to the velocity of light is simply equal to the velocity of light.

(v) It is impossible to send one signal with a velocity greater than the velocity of light.

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